

Duality of Banach Spaces

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Abstract

“Duality in mathematics is not a theorem, but a principle.” - *Michael Atiyah*

In this note we go through several important Banach spaces. We will give the definitions and some useful properties of the spaces. Furthermore, we will discuss the dualities of these spaces, which will establish connections among them.

1 l^p space

Definition 1.1. For a $p \in [1, \infty)$, we define $\ell^p := \{x = (x_1, x_2, \dots) : x_n \in \mathbb{F}, \|x\|_p < \infty\}$, where the norm is:

$$\|x\|_p := \left(\sum_{k=1}^{\infty} |x_k|^p \right)^{\frac{1}{p}}$$

Definition 1.2 (Hölder’s conjugate). We say $p, q \in (1, \infty)$ where $\frac{1}{p} + \frac{1}{q} = 1$ satisfy the *Hölder’s conjugate*.

Definition 1.3 (Dual Space [2]). The *dual* of a norm space X is $X^* = \{\theta : X \rightarrow \mathbb{F} : \theta \text{ is linear and bounded}\}$.

Definition 1.4 (Operator Norm [3]). Let $A : V \rightarrow W$ be a linear operator between normed spaces. The first four definitions are always equivalent by [4], and if in addition $V \neq \{0\}$ then they are all equivalent:

$$\|A\|_{\text{op}} = \inf\{c \geq 0 : \|Av\| \leq c\|v\| \text{ for all } v \in V\}$$

$$\begin{aligned}
&= \sup\{\|Av\| : \|v\| \leq 1 \text{ and } v \in V\} \\
&= \sup\{\|Av\| : \|v\| < 1 \text{ and } v \in V\} \\
&= \sup\{\|Av\| : \|v\| \in \{0, 1\} \text{ and } v \in V\} \\
&= \sup\{\|Av\| : \|v\| = 1 \text{ and } v \in V\} \text{ this equality holds if and only if } V \neq \{0\} \\
&= \sup\left\{\frac{\|Av\|}{\|v\|} : v \neq 0 \text{ and } v \in V\right\} \text{ this equality holds if and only if } V \neq \{0\}.
\end{aligned}$$

We skip the proof and remember these as a fact.

Theorem 1.1 (Hölder's inequality [1]). Let (x_n) and (y_n) be sequences in $\mathbb{R}$ such that $(\sum_{k=1}^{\infty} |x_n|^p)$ and $(\sum_{k=1}^{\infty} |y_n|^q)$ are convergent with p and q satisfy the Hölder's conjugate. Then:

$$\sum_{k=1}^{\infty} |x_n y_n| \leq \left(\sum_{k=1}^{\infty} |x_n|^p\right)^{1/p} \left(\sum_{k=1}^{\infty} |y_n|^q\right)^{1/q}$$

Theorem 1.2. The dual of ℓ^p space is ℓ^q , where p and q satisfy the Hölder's conjugate i.e $\frac{1}{p} + \frac{1}{q} = 1$ ($p, q \in (1, \infty)$), and we write it as $(\ell^p)^* \cong \ell^q$, i.e there is an isometric isomorphism between $(\ell^p)^*$ and ℓ^q .

Proof. Define a map

$$\begin{aligned}
T : \ell^q &\longrightarrow (\ell^p)^* \\
(Tx)(y) &:= \sum_{n=1}^{\infty} x_n y_n, \text{ where } x \in \ell^q \text{ and } y \in \ell^p
\end{aligned}$$

Well-defined: First observe

$$|(Tx)(y)| \leq \lim_{N \rightarrow \infty} \sum_{n=1}^N |x_n y_n|$$

and using the Hölder's inequality, we have

$$\lim_{N \rightarrow \infty} \sum_{n=1}^N |x_n y_n| \leq \|x\|_q \|y\|_p < \infty$$

so Tx is **bounded**, also since this map is defined by an absolute convergent series, so the **linearity** comes naturally. Hence we have the map T is **well-defined**.

Boundedness and linearity: Now if we fix a particular $x \in \ell^q$, then by the definition

of operator norm and Hölder's inequality:

$$\|Tx\|_{\ell^p \rightarrow \mathbb{F}} = \sup_{\|y\|_p=1} |(Tx)(y)| \leq \sup_{\|y\|_p=1} \|x\|_q \|y\|_p = \|x\|_q$$

and we can take the x out of the norm:

$$\|T\| = \sup_{\|x\|_q=1} \|Tx\|_{\ell^p \rightarrow \mathbb{F}} \leq \sup_{\|x\|_q=1} \|x\|_q = 1$$

therefore we have T is bounded for every x and y , and the linearity comes naturally from the series definition again.

Note this boundedness and linearity are different from these in well-defined, since here we are talking about $(Tx)(y) \in \mathbb{R}$, and in the previous part we are talking about $Tx \in (\ell^p)^*$.

Surjectivity:

First consider the case when $p, q \in (1, \infty)$. Let $\theta \in (\ell^p)^*$ and e_n be the standard sequence. Then define $x_n := \theta(e_n)$ and $x := (x_n)_{n \in \mathbb{N}}$, we claim $x \in \ell^q$ and $Tx = \theta$, consider the partial sum:

$$\begin{aligned} \sum_{n=1}^N |x_n|^q &= \sum_{n=1}^N x_n t_n, \quad t_n := \begin{cases} \frac{|x_n|^q}{x_n}, & x_n \neq 0, \\ 0, & x_n = 0, \end{cases} \\ &= \sum_{n=1}^N t_n \theta(e_n) = \theta \sum_{n=1}^N t_n e_n^1, \text{ since } \theta \text{ is linear.} \\ &\leq \|\theta\|_{\ell^p \rightarrow \mathbb{F}} \left\| \sum_{n=1}^N t_n e_n \right\|_p = \|\theta\|_{\ell^p \rightarrow \mathbb{F}} \left(\sum_{n=1}^N |t_n|^p \right)^{\frac{1}{p}} \\ &= \|\theta\|_{\ell^p \rightarrow \mathbb{F}} \left(\sum_{n=1}^N \left(\frac{|x_n|^q}{|x_n|} \right)^p \right)^{\frac{1}{p}} = \|\theta\|_{\ell^p \rightarrow \mathbb{F}} \left(\sum_{n=1}^N |x_n|^{(q-1)p} \right)^{\frac{1}{p}} \\ &= \|\theta\|_{\ell^p \rightarrow \mathbb{F}} \left(\sum_{n=1}^N |x_n|^q \right)^{\frac{1}{p}}, \text{ since } \frac{1}{p} + \frac{1}{q} = 1. \\ &\implies \left(\sum_{n=1}^N |x_n|^q \right)^{1-\frac{1}{p}} \leq \|\theta\|_{\ell^p \rightarrow \mathbb{F}} \\ &\implies \left(\sum_{n=1}^N |x_n|^q \right)^{\frac{1}{q}} \leq \|\theta\|_{\ell^p \rightarrow \mathbb{F}} \end{aligned}$$

Therefore $x \in \ell^q$.

For the second, for arbitrary $y \in \ell^p$:

$$\begin{aligned}(Tx - \theta)(y) &= (Tx - \theta) \lim_{N \rightarrow \infty} \sum_{n=1}^N y_n e_n = \lim_{N \rightarrow \infty} (Tx - \theta) \sum_{n=1}^N y_n e_n \\ &= \lim_{N \rightarrow \infty} \sum_{n=1}^N y_n (Tx - \theta) e_n\end{aligned}$$

and since $(Tx)(e_n) = x_n$ for each n , and $\theta(e_n) = x_n$ by definition, so we conclude the difference between Tx and θ is 0 for all $y \in \ell^p$, which means $Tx = \theta$.

(Special case: Consider $p = 1$ and $q = \infty$, $\theta \in (\ell^1)^*$, since θ is bounded, then $\sup_{n \in \mathbb{N}} |x_n| = \sup_{n \in \mathbb{N}} |\theta(e_n)| \leq M$ for some constant M , so $x \in \ell^\infty$ by definition. Then use the same technique above to conclude $Tx = \theta$.)

Isometry: It is sufficient to check $\|Tx\|_{\ell^p \rightarrow \mathbb{F}} = \|x\|_q$, and we already know $\|Tx\|_{\ell^p \rightarrow \mathbb{F}} \leq \|x\|_q = \left(\sum_{n=1}^{\infty} |x_n|^q \right)^{\frac{1}{q}} \leq \|\theta\|_{\ell^p \rightarrow \mathbb{F}} = \|Tx\|_{\ell^p \rightarrow \mathbb{F}}$, and this also implies the injectivity.

Therefore we find an isometric isomorphism from $(\ell^p)^*$ to ℓ^q which means $(\ell^p)^* \cong \ell^q$. □

¹Note this sum should be a sequence, and since it has only finitely many non-zero terms, so it has a norm in ℓ^p actually.

2 *ba* space

Definition 2.1 (Algebra on a set [5]). An **Algebra** $\mathcal{A}$ on the set Ω is a collection of subsets of Ω st:

1. $\emptyset \in \mathcal{A}$,
2. $\forall A, B \in \mathcal{A}, A \cup B \in \mathcal{A}$ and $A^c \in \mathcal{A}$,

Note after the second we immediately have:

3. $A \cap B = (A^c \cup B^c)^c \in \mathcal{A}$,
4. $A/B = A \cap B^c \in \mathcal{A}$,
5. $\Omega \in \mathcal{A}$.

Definition 2.2 (Finitely additive measure or Charge[5]). Given a set Ω and its algebra $\mathcal{A}$, then the **charge** of $(\Omega, \mathcal{A})$ is a function

$$\mu : \mathcal{A} \longrightarrow \mathbb{R}$$

s.t:

1. $\mu(\emptyset) = 0$,
2. $\mu(\bigsqcup_{i=1}^n A_i) = \sum_{i=1}^n \mu(A_i)$ for any disjoint finite subsets $A_1, \dots, A_n \in \mathcal{A}$.

Example 2.3. 1. Let $X = \{1, 2, 3\}$, and an algebra on it is:

$$\mathcal{A} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}\}$$

then we define the charge μ by:

$$\mu(\{1\}) := 1, \mu(\{2\}) := 2, \mu(\{3\}) := 3 \text{ and } \mu(A) := \sum_{i \in A} \mu(i).$$

we can easily check this μ satisfies the conditions for charge.

Definition 2.4 (Bounded [7]). A charge μ is called **bounded** if

$$\sup_{\substack{\text{finite} \\ \text{partition } \{A_i\}}} \sum |\mu(A_i)| < \infty.$$

Note: Finite partition $\{A_i\}$ is the union of some finite subsets A_i which in the algebra $\mathcal{A}$ on Ω s.t they are pairwise disjoint.

Definition 2.5 (Total variation[6]). The total variation of μ on A is defined as

$$|\mu|(\Omega) = \sup_{\substack{\text{finite} \\ \text{partition } \{A_i\}}} \sum |\mu(A_i)|$$

Fact: the total variation is the smallest positive charge λ such that $\lambda(\Omega) \geq |\mu(A)|$ for every $A \subset \Omega$.

Definition 2.6 (*ba space*). The *ba space* $ba(\Omega, \mathcal{A})$ is the norm space which contains all bounded charges of $(\Omega, \mathcal{A})$, and the norm of the space is defined to be:

$$\|\mu\| := \sup_{\substack{\text{finite} \\ \text{partition } \{A_i\}}} \sum |\mu(A_i)|.$$

Note: we can check that this “norm” is actually a norm:

1. It is clear that for every charge μ , $\sup \sum |\mu(A_i)| \geq 0$. If $\sup \sum |\mu(A_i)| = 0$, consider the partition $\{A, \Omega \setminus A\}$. Then

$$|\mu(A)| + |\mu(\Omega \setminus A)| = 0 \implies \mu(A) = 0 \quad \text{and} \quad \mu(\Omega \setminus A) = 0.$$

Since A is arbitrary, we get $\mu = 0$.

2. Homogeneity and the triangle inequality hold by straightforward arguments.

Proposition 2.7 (Completeness of *ba space*). $(\Omega, \mathcal{A})$ is complete (i.e. a Banach space).

Proof. Suppose we have a Cauchy sequence $\{\mu_n\} \subset ba(\Omega, \mathcal{A})$, which means:

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall n, m > N \implies \|\mu_n - \mu_m\| < \epsilon.$$

This implies: if we choose an arbitrary subset $A \in \mathcal{A}$, since $\{A, \Omega \setminus A\}$ is a partition of Ω , we have

$$|\mu_n(A) - \mu_m(A)| \leq \|\mu_n - \mu_m\| < \epsilon$$

by the definition of the norm. Hence $\{\mu_n(A)\}$ is a Cauchy sequence in $\mathbb{R}$, so it converges to some limit by completeness of $\mathbb{R}$, denoted $\mu(A)$. We claim that μ is the limit of $\{\mu_n\}$ in $ba(\Omega, \mathcal{A})$.

First we need to check this limit is actually in $(\Omega, \mathcal{A})$ i.e it is finitely additive and bounded:

Finitely additive: Let $A, B \in \mathcal{A}$ be disjoint. Then

$$\mu_n(A \cup B) = \mu_n(A) + \mu_n(B).$$

Taking limits as $n \rightarrow \infty$,

$$\mu(A \cup B) = \lim_{n \rightarrow \infty} \mu_n(A \cup B) = \lim_{n \rightarrow \infty} (\mu_n(A) + \mu_n(B)) = \lim_{n \rightarrow \infty} \mu_n(A) + \lim_{n \rightarrow \infty} \mu_n(B) = \mu(A) + \mu(B).$$

By induction, μ is finitely additive on $\mathcal{A}$.

Boundedness: We show $\|\mu\| < \infty$. Observe

$$\|\mu\| = \sup_{\substack{\text{finite} \\ \text{partition } \{A_i\}}} \sum |\mu(A_i)| = \sup_{\substack{\text{finite} \\ \text{partition } \{A_i\}}} \sum \left| \lim_{n \rightarrow \infty} \mu_n(A_i) \right|.$$

Since each partition is finite, we can exchange limit and finite sum:

$$\sum \left| \lim_{n \rightarrow \infty} \mu_n(A_i) \right| = \lim_{n \rightarrow \infty} \sum |\mu_n(A_i)| \leq \lim_{n \rightarrow \infty} \left(\sum |\mu_n(A_i)| \right).$$

Because $\{\mu_n\}$ is a Cauchy sequence in a normed space, it is bounded. Hence there exists some constant C such that

$$\sum |\mu_n(A_i)| \leq C \quad \text{for all finite partitions } \{A_i\},$$

and therefore $\|\mu\| \leq C < \infty$. Thus μ is bounded.

Converge to μ :

since μ_n is Cauchy, so $\forall \epsilon > 0, \exists N$ st $\forall n, m > N$:

$$\begin{aligned} \|\mu_n - \mu\| &= \sup_{\substack{\text{finite} \\ \text{partition } \{A_i\}}} \sum |\mu_n(A_i) - \mu(A_i)| \\ &\leq \sup_{\substack{\text{finite} \\ \text{partition } \{A_i\}}} \left(\sum |\mu_n(A_i) - \mu_m(A_i)| + \sum |\mu_m(A_i) - \mu(A_i)| \right) \\ &< \epsilon \end{aligned}$$

hence $\lim_{n \rightarrow \infty} \mu_n = \mu$ in $ba(\Omega, \mathcal{A})$. □

Definition 2.8. Given a set $A \subset X$, then $\mathbb{1}_A$ is the map:

$$\mathbb{1}_A : X \longrightarrow \{0, 1\}, \quad \mathbb{1}_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases}$$

Note that when $X = \mathbb{N}$ and $\mathbb{1}_A(n)$ is actually a sequence defined by A .

Definition 2.9. Let $(\Omega, \mathcal{A})$ be a measurable space² and $\mu \in \text{ba}(\Omega, \mathcal{A})$.

- (1) $\mu^+ : \mathcal{A} \rightarrow \mathbb{R} \quad \mu^+(F) = \sup\{\mu(E) : E \subseteq F, E \in \mathcal{A}\}.$
- (2) $\mu^- : \mathcal{A} \rightarrow \mathbb{R} \quad \mu^-(F) = -\inf\{\mu(E) : E \subseteq F, E \in \mathcal{A}\}.$
- (3) $\mu : \mathcal{A} \rightarrow \mathbb{R} \quad \mu = \mu^+ + \mu^-.$

We can check μ is actually the total variation of the ba space we defined above.

Proposition 2.10. We can easily check both μ^+ and $\mu^- \in \text{ba}(\Omega, \mathcal{A})$, and they are all positive charge.

Definition 2.11. [7] Let $\mu \in \text{ba}(\Omega, \mathcal{P}(\Omega))$ be a charge. First suppose μ is positive.

- Let $a = \sum_{n=1}^N \alpha_n \mathbb{1}_{A_n}$ be a bounded sequence ($\alpha_n \geq 0$) with $A_1, A_2, \dots, A_N \subset \Omega$ pairwise disjoint. Define

$$\int_{\Omega} a \, d\mu = \sum_{n=1}^N \alpha_n \mu(A_n)$$

Now suppose μ is a general finitely additive measure in $\text{ba}(\Omega, \mathcal{P}(\Omega))$.

- Write $\mu = \mu^+ - \mu^-$. Define for all $x \in L^\infty$,

$$\int_{\Omega} x \, d\mu = \int_{\Omega} x \, d\mu^+ - \int_{\Omega} x \, d\mu^-$$

Theorem 2.1. For a countable set Ω , the dual of ℓ^∞ is $\text{ba}(\Omega, \mathcal{A})$.

Proof. For $A \in \Omega$, define a map

$$T : (\ell^\infty)^* \longrightarrow \text{ba}(\Omega, \mathcal{A})$$

$$T(\theta) = \mu_\theta$$

$$\text{where } \mu_\theta : \mathcal{A} \longrightarrow \mathbb{R}, \mu_\theta(A) = \theta(\mathbb{1}_A)$$

Not like the l^p , we don't need to worry about the well-defined, since the map T is define by a linear and bounded map θ already.

Note if Ω is a countable set, then we can enumerate the x with the natural number n to be its index and think of $\mathbb{1}_A$ as a sequence.

²the measurable space is a set with a σ -algebra on the set. The definition of σ -algebra is similar as algebra but it is stronger since it talks about countable instead of finitely, but we won't go deeper on it.

Boundedness and finitely additive : Check the two conditions by linearity of θ :
 $\mu_\theta(\emptyset) = \theta(\mathbf{1}_\emptyset) = \theta(0) = 0$. $\mu_\theta(\bigsqcup_{i=1}^n A_i) = \theta(\bigsqcup_{i=1}^n A_i) = \sum_{i=1}^n \theta(A_i) = \sum_{i=1}^n \mu_\theta(A_i)$
Consider

$$|\mu_\theta(A)| = |\theta(\mathbf{1}_A)| \leq \|\theta\|_{(\ell^\infty)^*} \|\mathbf{1}_A\|_\infty \leq \|\theta\|_{(\ell^\infty)^*} < \infty$$

so μ_θ is bounded.

Surjectivity : Let $\mu \in ba(\Omega, \mathcal{A})$, define:

$$\theta : \ell^\infty \longrightarrow \mathbb{R} \quad \theta(x) = \int_\Omega x d\mu$$

we claim $T(\theta) = \mu$, first show this θ is well-defined: linearity is clear by the definition of integral, for the boundedness, let $A_1, A_2, \dots, A_n$ be a finitely partition of Ω , and consider

$$\begin{aligned} |\theta(x)| &= \left| \int_\Omega x d\mu \right| = \left| \int_\Omega x d\mu^+ - \int_\Omega x d\mu^- \right| \leq \int_\Omega |x| d\mu^+ + \int_\Omega |x| d\mu^- \\ &\leq \int_\Omega \|x\|_\infty d\mu^+ + \int_\Omega \|x\|_\infty d\mu^- \leq \|x\|_\infty \sum_{n=1}^N \mu^+(A_i) + \|x\|_\infty \sum_{n=1}^N \mu^-(A_i) \\ &= \|x\|_\infty \mu^+(\Omega) + \|x\|_\infty \mu^-(\Omega) \\ &\leq \|x\|_\infty \|\mu\|_{ba} < \infty \end{aligned}$$

and $\forall A \subset \Omega$, we find: $T(\theta)(A) = \mu_\theta(A) = \theta(\mathbf{1}_A) = \int_\Omega \mathbf{1}_A d\mu = \mu(A)$.

Isometry : We want to show $\|\theta\|_{(\ell^\infty)^*} = \|\mu_\theta\|_{ba}$ for all $\theta \in (\ell^\infty)^*$, we first prove the one side inequality: let $x \in \ell^\infty$ st $\|x\|_\infty \leq 1$ and let $a \in \ell^\infty$ st it is bounded, so we can write $a = \sum_{n=0}^N \alpha_n \mathbf{1}_{A_i}$, where A_i are some pairwise disjoint subset and α_i is some real constants. Since the set collects all such a is dense in ℓ^∞ (further more $|a_n| \leq |x_n|$ which can be easily shown), so for the chosen x and arbitrary positive ϵ , there is an a st:

$$\begin{aligned} \|x - a\|_\infty &< \frac{\epsilon}{\|\theta\|_{(\ell^\infty)^*}} \\ \implies \|\theta\|_{(\ell^\infty)^*} \|x - a\|_\infty &< \epsilon \implies |\theta(x - a)| < \epsilon \implies |\theta(x)| - |\theta(a)| < \epsilon \\ \implies |\theta(x)| &< |\theta(a)| + \epsilon \end{aligned}$$

then we consider

$$|\theta(a)| = \left| \theta\left(\sum_{n=1}^N \alpha_n \mathbf{1}_{A_n}\right) \right| = \left| \sum_{n=1}^N \alpha_n \theta(\mathbf{1}_{A_n}) \right| \leq \sum_{n=1}^N |\alpha_n| |\theta(\mathbf{1}_{A_n})| \leq \sum_{n=1}^N |\mu_\theta(A_n)| \leq \|\mu_\theta\|_{ba}$$

so we have $|\theta(x)| < \|\mu_\theta\|_{ba} + \epsilon$, by the definition of operator norm and we have chosen x with $\|x\|_\infty \leq 1$, so have $\|\theta\|_{(\ell^\infty)^*} \leq \|\mu_\theta\|_{ba}$.

For the other side, let $A_1, A_2, \dots, A_N \subseteq \Omega$ be pairwise disjoint. Take $\delta_n \in \{-1, 1\}$ st $|\mu(A_i)| = \delta_n \mu(A_n)$. Define $x = \sum_{n=1}^N \delta_n \mathbf{1}_{A_n}$, then we have $\|x\|_\infty \leq 1$. Then

$$\sum_{n=1}^N |\mu_\theta(A_n)| = \sum_{n=1}^N \delta_n \mu_\theta(A_n) = \sum_{n=1}^N \delta_n \theta(\mathbf{1}_{A_n}) = \theta\left(\sum_{n=1}^N \delta_n (\mathbf{1}_{A_n})\right) = \theta(x) \leq |\theta(x)| \leq \|\theta\|_{(\ell^\infty)^*}. \quad (10)$$

Therefore we have the equation which proves the isometry (also injectivity).

Finally we conclude $(l^\infty)^* \cong ba(\Omega, \mathcal{A})$ for countable Ω . □

3 Structure of $(\ell^\infty)^*$

Theorem 3.1 (Hahn-Banach Theorem[6]). Let X be a real or complex normed linear space, let $A \subset X$ be a linear subspace, and let $\theta \in A^*$ be a bounded linear functional on A . Then there exists a linear functional $\phi \in X^*$ that extends θ and satisfies $\|\phi\|_{X^*} = \|\theta\|_{A^*}$.

Note: The proof of Hahn-Banach Theorem can be found in many textbooks and it is not the key of our study so we will skip the proof here.

Definition 3.1 (Natural Embedding[8]). Given a linear normed space X , its bidual X^{**} naturally exist, and there exists a mapping:

$$J : X \longrightarrow X^{**}$$

$$J(x) = J_x, x \in X$$

where $J_x \in X^{**}$ and is given by

$$J_x : X^* \longrightarrow \mathbb{R}$$

$$J_x(x^*) = x^*(x), x^* \in X^*$$

moreover $\|x\|_X = \|J_x\|_{X^{**}}$.

This J is called the natural embedding of X to X^{**} and the image of J is denoted as $\hat{X}$.

Proof.

$$\begin{aligned} |J_x(x^*)| &= |x^*(x)| = |x^*(x)| \|x\|_X \frac{1}{\|x\|_X} = |x^*\left(\frac{x}{\|x\|_X}\right)| \|x\|_X \quad \text{since } x^* \text{ is linear} \\ &\leq \|x^*\|_{X^*} \|x\|_X \quad \text{by the definition of operator norm} \end{aligned}$$

so we have:

$$\|J_x\|_{X^{**}} = \sup_{\substack{\|x^*\|_{X^*}=1 \\ x^* \in X^*}} |J_x(x^*)| \leq \sup_{\substack{\|x^*\|_{X^*}=1 \\ x^* \in X^*}} \|x^*\|_{X^*} \|x\|_X = \|x\|_X$$

On the other side, define $X_1 = \{x \in X : \|x\|_X = 1\}$, then for $\|\cdot\|_{X_1} \in X_1^*$ and Hahn-Banach Extension, we have a corresponding $y^* \in X^*$ s.t

$$(1) \quad y^*(x_1) = \|x_1\|_{X_1} = 1, \forall x_1 \in X_1$$

$$(2) \quad \|y^*\|_X = \|\|\cdot\|_{X_1}\|_{X_1^*} = \sup_{\substack{\|x_1\|_{X_1}=1 \\ x_1 \in X_1}} \|x_1\|_{X_1} = 1$$

therefore

$$\begin{aligned} \|J_x\|_{x^{**}} &= \sup_{\substack{\|x^*\|_{X^*}=1 \\ x^* \in X^*}} |J_x(x^*)| \geq |J_x(y^*)| \quad \text{from (2)} \\ &= |y^*(x)| = \|x\|_X, \quad \forall x \in X \end{aligned}$$

Combine both side, we have $\|J_x\|_{x^{**}} = \|x\|_X$. □

Proposition 3.2. For every linear norm space X , we have $X \subset X^{**}$.

Proof. From the existence of natural embedding, $X \cong J(X) \subset X^{**}$.

Also note that in general we have the strict inclusion since embedding may not be surjective.

Below is a classical example which tells why we can't have Theorem 1.2. true in the other direction. □

Example 3.3. $(\ell^\infty)^* \not\cong \ell^1$

Proof. All the elements in ℓ^1 should be understand as a linear map $T_x(y) = \sum_{n=1}^{\infty} x_n y_n$ where $(x_n) \in \ell^1$ and $(y_n) \in \ell^\infty$.

Hence we need to find one bounded linear map in ℓ^∞ which is not in ℓ^1 i.e it can't be written as the above given form:

Consider the Banach space $c \leq \ell^\infty$ which is a countable space and contains all the convergent sequences. Then we define a map $f : c \rightarrow \mathbb{R}$ by $f(x) = \lim_{n \rightarrow \infty} x_n$, it is bounded and linear by fundamental algebras, and now using Hahn-Banach Theorem, this map can be extended to ℓ^∞ by $g : \ell^\infty \rightarrow \mathbb{R}$.

Now suppose this g is in the summation form, for every $(x_n) \in \ell^\infty$, we would have

$$x_n = g(e_n) = f(e_n) = 0$$

for every $n \in \mathbb{N}$, this means g is identically zero, but since $g(1, 1, \dots) = 1$, so contradiction. □

³note the dual of ℓ^1 is ℓ^∞

Definition 3.4 (Reflexive). If the natural embedding of X to X^{**} is bijective i.e $\widehat{X} = X^{**}$, we call X reflexive.

Example 3.5. By Example 3.3, ℓ^1 is not reflexive.

Lemma 3.6. The closed subspace of a reflexive Banach space is also reflexive.

Proof. Suppose Y is a closed subspace of a reflexive Banach space X , then consider the natural embedding:

$$J : Y \longrightarrow Y^{**}$$

$$J(y) = J_y, \quad y \in Y$$

choosing any $\theta \in Y^{**}$, we must show $\exists y \in Y$, s.t $J_y = \theta$.

First define $x^{**} \in X^{**}$, s.t $x^{**}(x^*) = \theta(x^*|_Y)$ ⁴, then by the reflexivity of X , $\exists x \in X$, s.t $J_x = x^{**}$, and we claim this J_x is exactly the J_y we want, this means the x should be the element of Y . Assume it is not and Y is a proper subspace of X , then by Hahn-Banach Theorem, we know $\exists \phi \in X^*$, s.t $\phi(x) = 1$ and $\phi|_Y = 0$ (note Y is closed), so $x^{**}(\phi) = J_x(\phi) = \phi(x) = 1$, but on the other side $x^{**}(\phi) = \theta(\phi|_Y) = 0$, contradiction. If $Y = X$, it is trivial.

Left to prove $J_x = \theta$: for any $y^* \in Y^*$, let $x^* \in X^*$ be the Hahn-Banach extension of it.

$$J_x(y^*) = y^*(x) = x^*(x) = J_x(x^*) = x^{**}(x^*) = \theta(x^*|_Y) = \theta(y^*).$$

□

Proposition 3.7. Let X be a Banach space, then X is reflexive iff X^* is reflexive.

Proof. $\implies$ For the natural embedding:

$$J^* : X^* \longrightarrow X^{***}$$

$$J^*(x^*) = J_{x^*}^*, \quad x^* \in X^*$$

We must show choosing any $\theta \in X^{***}$, $\exists x^* \in X^*$, s.t $J_{x^*}^* = \theta$, that is for every $y^{**} \in X^{**}$, $J_{x^*}^*(y^{**}) = \theta(y^{**})$.

⁴ $x^*|_Y$ is the linear map x^* restricted in Y .

Consider the map $\theta \circ J$, we claim it is the x^* we need.
Indeed, this map is in X^* .

$$J_{\theta \circ J}^*(y^{**}) = y^{**}(\theta \circ J) \text{ by definition of natural embedding}$$

by the reflexivity of X , we know $\exists x \in X$, s.t $J_x = y^{**}$, so

$$\begin{aligned} y^{**}(\theta \circ J) &= J_x(\theta \circ J) = (\theta \circ J)(x) \\ &= \theta(J(x)) = \theta(J_x) \\ &= \theta(y^{**}) \end{aligned}$$

$\Leftarrow$ First note that the image J_x (which is $\widehat{X}$ as the notation above) of the natural embedding $J : X \longrightarrow X^{**}$ is isometric isomorphic to X , since J preserves the norm and it is indeed surjective.

Hence we only need to prove $\widehat{X} \cong X^{**}$. Consider a sequence x_n in X (then $J(x_n)$ is naturally a sequence in $\widehat{X}$), now suppose $J(x_n) \longrightarrow y$, since J preserves the norm, $\|x_n - x_m\| = \|J(x_n) - J(x_m)\|$, so x_n is Cauchy, and it has a limit x in X , since J is continuous so we have $J(x_n) \longrightarrow J(x) = y$, which means $y \in \widehat{X}$, so $\widehat{X}$ is closed. Also we know X^{**} is reflexive, so by Lemma 3.6., $X \cong \widehat{X}$ is also reflexive. $\square$

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